Helfferich, F. G., and G. Klein, *Multicomponent Chromatography: Theory of Interference*, Marcel Dekker, New York, 52 (1970).

Holm, L. W., "Design, Performance and Evaluation of the Uniflood Micellar-Polymer Process-Bell Creek Field," SPE 11196, Annual Tech. Conf. and Exhibition of the Soc. of Petrol. Engineers, New Orleans (Sept. 26–29, 1982).

Lake, L. W., and H. J. Hill, "Cation Exchange in Chemical Flooding: Part 3—Experimental," *Soc. Petrol. Engrs. J.*, 18(6), 445 (1978).

Lake, L. W., and G. W. Pope, "Status of Micellar-Polymer Field Tests," *Petrol. Eng. Int.*, **51**(13), 38 (1979).

Mysels, K. J., and R. J. Otter, "Thermodynamic Aspects of Mixed Micelles—Application of an Empirically Established Equilibrium," *J. Colloid Sci.*, **16**, 474 (1961).

Scamehorn, J. F., R. S. Schechter, and W. H. Wade, "Adsorption of Surfactants on Mineral Oxide Surfaces from Aqueous Solutions. I. Isomerically Pure Anionic Surfactants," *J. Colloid and Interface Sci.*, 85(2), 463 (1982a); "II. Binary Mixture of Anionic Surfactants," *J. Colloid and Interface Sci.*, 85(2), 476 (1982b).

Shinoda, K., *Colloidal Surfactant*, Academic Press, New York, 65 (1963).

Somasundaran, P., and D. W. Fuerstenau, "Mechanisms of Alkyl Sulfonate Adsorption at the Alumina-Water Interface," *J. Phys. Chem.*, **70**(1), 90 (1966).

Trogus, F. J., et al., "Adsorption of Mixed Surfactant Systems," *J. Petrol. Technol.*, **31**, 769 (1979a).

Trogus, F. J., R. S. Schechter, and W. H. Wade, "A New Interpretation of Adsorption Maxima and Minima," *J. Colloid and Interface Sci.*, **70**, 293 (1979b).

Manuscript received Aug. 15, 1983; revision received Jan. 20, 1984, and accepted Jan. 28.

Synthesis of Control Structures by Singular Value Analysis: Dynamic Measures of Sensitivity and Interaction

A strategy for synthesizing regulatory control structures is developed within the framework provided by singular value decomposition (SVD). Quantitative measures of interaction and sensitivity for the nodes, as well as for the entire system, are established with this method. The approach enables analysis of the control structure over a range of frequencies which are of practical importance for a particular processing unit, thus insuring that both static and dynamic effects are encompassed. In addition, this method allows the designer to identify the modeling aspects which are important in determining the performance of the synthesized control structures. A series of examples, presented in order of increasing complexity, demonstrate the usefulness and versatility of this procedure. The technique is implemented as an interactive computer package.

HENRY LAU, JESÚS ALVAREZ,
and K. F. JENSEN

Department of Chemical Engineering
and Materials Science
University of Minnesota
Minneapolis, MN 55455

SCOPE

The design of a regulatory control structure for a chemical process is one of the most challenging tasks confronting the control engineer. It has been suggested (Foss, 1973; Lee and Weekman, 1976; Stephanopoulos, 1982) that the relevant problem in chemical engineering process control is not the development of more sophisticated algorithms, but rather the establishment of a structural framework for selecting the manipulated and measured variables and linking them appropriately. In industrial practice the loop pairing is often based on the experience and intuition of the process control engineer. Many techniques have been proposed to rationalize the selection of loop pairings. Bristol's relative gain array (Bristol, 1966), which indicates the proper pairings for minimum static interactions among control loops, has been popular because it is easy to use and only requires knowledge of steady-state process gains. However, stronger interactions may exist during transients, and these may dictate a different control structure than that predicted by the static analysis (Tung and Edgar, 1977). Consequently, measures of process interaction have been proposed which include dynamic effects (Witcher and McAvoy, 1977; Tung and Edgar, 1977; Gagnepain and Seborg, 1982). However, the simple criteria may not be as reliable as computer-aided design packages such as the inverse Nyquist array or the characteristic loci (Ray, 1982).

In this paper we describe a control structure synthesis strategy based on the singular values (also known as principal gains) of the open-loop transfer function. The magnitude of the singular values measures the sensitivity of multiinput/multioutput (MIMO) systems in the same manner as the amplitude ratio is employed in single-input/single-output (SISO) systems. In addition, new and quantitative measures of interactions among control loops are developed. In order to encompass both static and dynamic features, the analysis is carried out over a range of frequencies of practical significance for a given process. Moreover, the analysis is combined with the design of structural compensators which minimize interactions among control loops over a range of frequencies characteristic of major process disturbances. The methodology is illustrated with examples, and particular emphasis is placed on the structural and physical interpretations of the process analysis and of the resulting control structure.

Correspondence concerning this paper should be addressed to K. F. Jensen.

Henry Lau is presently at Shell Development Company, Westhollow Research Center, Houston, TX 77001.

Jesús Alvarez is presently at Universidad Autonoma Metropolitana (Iztapalapa), Departmento de Ingenieria, 09340 Mexico D.F.

CONCLUSIONS AND SIGNIFICANCE

A systematic approach to the synthesis of regulatory process control structures is formulated by employing a singular value decomposition (SVD) technique. The approach provides new quantitative measures of interaction and sensitivity for the nodes as well as for the entire system. These measures are implemented in an interactive computer package which allows the designer to systematically select pairings of measured and manipulated variables while taking practical considerations into account.

The analysis is performed over a frequency range which is of practical importance for the particular chemical process such that both static and dynamic aspects are considered. The analogies between sensitivity measures of SISO and MIMO are exploited and the results of the analysis are represented graphically in plots similar to Bode diagrams. An additional important feature of the SVD strategy is its ability to identify modeling aspects, such as model mismatch, which affect the performance of the resulting process control structure. In addition, the strategy shows whether or not a structural decoupler will be effective in minimizing interactions among loops. The compensator can readily be designed for the range of frequencies most likely to affect the process. Four examples, including two distillation systems, illustrate the usefulness of the procedure and the variety of systems which can be analyzed by the SVD methodology.

INTRODUCTION

The development of control systems for chemical plants can be divided into three parts: formulation of control objectives, selection of control configuration, and determination of the appropriate control law. The latter part has received a great deal of attention and a large body of theory exists for designing optimal controllers for unit processes. The first two parts have only recently attracted significant interest (Morari, 1981, 1982; Stephanopoulos, 1982). This development has been spurred by the need for efficient, reliable control structures for the highly interconnected chemical plants which have been constructed as a result of the shortages in raw materials and energy. These plants have to show high performance within tight process specifications, as well as satisfy safety and environmental regulations. Consequently, the synthesis of the control structure is very important. In particular, the choice of manipulated and measured variables and the pairing of these variables in the control loop structure need to be considered.

The question of control loop pairings has received considerable attention. Bristol's (1966) relative gain array (RGA), which indicates the proper pairings for minimum static interactions among control loops, has been popular because it is easy to use and only requires knowledge of the steady-state process gains. Nevertheless, stronger control loop interactions may exist during transients and dictate an entirely different control structure than that predicted by the steady-state analysis (Tung and Edgar, 1977). Therefore, the relative gain array concept has been extended to include dynamic effects (Tung and Edgar, 1977; Witcher and McAvoy, 1981; Gagnepain and Seborg, 1982). However, these simple criteria may not be as reliable as computer-aided design packages such as the inverse Nyquist array or the characteristic loci (Ray, 1982).

In this paper we propose a singular value decomposition (SVD) technique as a systematic and rigorous approach to the problem of loop selection in control structures. Singular values (also known as principal gains) of the transfer function matrix may be used to evaluate stability margins for multiinput/multioutput (MIMO) systems in the same manner as the amplitude ratio is used in single-input/single-output (SISO) systems (Doyle and Stein, 1981). Smith et al. (1981a,b) have adapted SVD to the loop selection in a steady-state system. However, no measure of interaction or systematic search procedure is considered. Morari (1982) used the SVD to quantify the control performance attainable in a process and interpreted implementability and sensitivity of the plant, concepts which quantify the resiliency of the plant, in terms of the norms of the transfer function operator. It is interesting that two problems at opposite ends of the hierarchy in process design and control structure synthesis emerge closely related after the appropriate analysis. In the present work we extend the SVD technique to the frequency domain such that the analysis may be carried out over a range of frequencies which are of practical significance for a given process unit. In addition, we define new measures for interaction and sensitivity of the process nodes as well as for the entire process. These measures are incorporated in an interactive computer package which allows a systematic analysis of a given process. Moreover, the analysis will show whether or not it is feasible to construct a compensator for the control structure to minimize interactions among the control loops. The compensator will be determined explicitly for a range of frequencies by following a procedure originally suggested by MacFarlane and Kouvaritakis (1977). In addition, we show that the SVD strategy enables us to identify modeling aspects, such as model mismatch, which affect the performance of the resulting control structure. The overall procedure is illustrated by four examples, including two distillation systems.

ANALYSIS OF CHARACTERISTICS OF THE CONTROL STRUCTURE SYNTHESIS PROBLEM

We consider a general linear multiinput/multioutput system which in the time domain is represented by the following vector equations:

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where dim $x = r$; dim $A = r \times r$; dim $u = n$, dim $B = r \times n$; dim $y = m$, dim $C = m \times r$; and $x(0) = 0$. By taking Laplace transforms of Eqs. 1 and 2, the input-output relation is expressed as,

$$y(s) = C(sI - A)^{-1} B u(s) \tag{3}$$

$$\equiv G(s)\, u(s) \tag{4}$$

where $G(s)$ is the $m \times n$ transfer function matrix which can be obtained either from a mechanistic model description or from some experimental tests. It is important to note that the number of available measured variables can differ from the number of manipulated variables, i.e., $m \neq n$. This is generally the case for chemical processes, in particular, large-scale systems. Since we aim to link the measured and manipulated variables in some optimal manner we must necessarily include all variables in the analysis.

As a first attempt to design a control configuration we could choose to synthesize a multiloop control system. This the conventional approach, is attractive because it is relatively easy to implement. However, it is well-known that interactions between loops may lead to tuning and stability problems. The key decision in this simple multiloop control is the proper pairing of measured and manipulated variables. In some instance there may exist natural interactions within the system which should be exploited by the proper combination of variables (Foss, 1973; Morari, 1980). Moreover, the performance of the control system should be satisfactory over the frequency band of characteristic disturbances. Thus the strategy to synthesize a control configuration should include a frequency domain analysis, particularly emphasizing the frequency spectrum of disturbances which affect the system.

In subsequent sections we will use SVD to develop a systematic control structure synthesis strategy. Relevant concepts such as natural loop structure, structural compensators, and gain compensators will be developed and their positions within the strategy will be identified.

SYSTEM ANALYSIS BY SINGULAR VALUE DECOMPOSITION

Singular value decomposition is a promising tool in the structural analysis of multivariable systems. The method is a powerful and computationally efficient tool for analyzing matrix systems (Forsythe, 1977). Currently, SVD has applications in systems engineering (Doyle and Stein, 1981; Safanov et al., 1981; Cruz, et al., 1981; Postlethwaite et al., 1981). These studies are based upon particular feedback controllers and consider some form of closed loop matrix operators. Since we are primarily interested in the control structure rather than in the actual controller design, we base our analysis on the open loop transfer function. The approach provides insights into important closed loop system properties: stability (Postlewaite et al., 1981), sensitivity (Weber, 1972), and invertibility (Morari, 1981). We shall further elaborate on these concepts in subsequent sections.

Algebraic Preliminaries

Application of SVD to the $m \times n$ transfer function matrix $G(s)$ leads to the equation (Noble and Daniel, 1977)

$$G(s) = Z(s)\,\Lambda(s)\,V^+(s) \tag{5}$$

where

$$\Lambda(s) = \begin{array}{cc} \left[\begin{array}{c|c} \Delta(s) & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} q \\ m-q \end{array} \\ \begin{array}{cc} q & \quad n-q \end{array} & \end{array}$$

$$q = \text{rank } G(s) \leq \min(m,n), \tag{5a}$$

and $(\cdot)^+$ denotes transposition. $\Delta(s)$ is a diagonal matrix whose entries are the singular values of $G(s)$; singular values are the positive square roots of the eigenvalues of the hermitian matrix $G^+(s)G(s)$ or $G(s)G^+(s)$. $Z(s)$ is a $m \times m$ matrix which is composed of the eigenvectors of the matrix $G(s)G^+(s)$, and $V(s)$ is a $n \times n$ matrix which is composed of the eigenvectors of $G^+(s)G(s)$. This decomposition implies that $(m - q)$ measurements and $(n - q)$ manipulated variables can be deleted without altering the input-output accessibility and manipulability of the system. However, a preliminary system analysis should include all the input and output variables.

Geometric Interpretations

Suppose $\sigma_1(s), \sigma_2(s), \ldots, \sigma_q(s)$ are distinct singular values of $G(s)$, then $Z(s)$ and $V(s)$ can be partitioned:

$$Z(s) = [z_1(s) \vdots z_2(s) \vdots \cdots \vdots z_q(s) \vdots z_{q+1}(s), \ldots, z_m(s)] \tag{6}$$

$$V(s) = [v_1(s) \vdots v_2(s) \vdots \cdots \vdots v_q(s) \vdots v_{q+1}(s), \ldots v_n(s)] \tag{7}$$

where $z_i(s)$ and $v_i(s)$ $(i = 1, \ldots, q)$ are the singular decomposition vectors which correspond to the ith singular value and $z_j(s)$ and $v_j(s)$ $(j = q + 1, \ldots, n)$ are the remaining decomposition vectors which correspond to the zero singular values.

An alternative expression for Eq. 5 is to write $G(s)$ as a sum of dyads,

$$G(s) = \sum_1^q \sigma_i(s)\, z_i(s)\, v_i^+(s) \tag{8}$$

$$\equiv \sum_1^q \sigma_i(s)\, W_i(s) \tag{9}$$

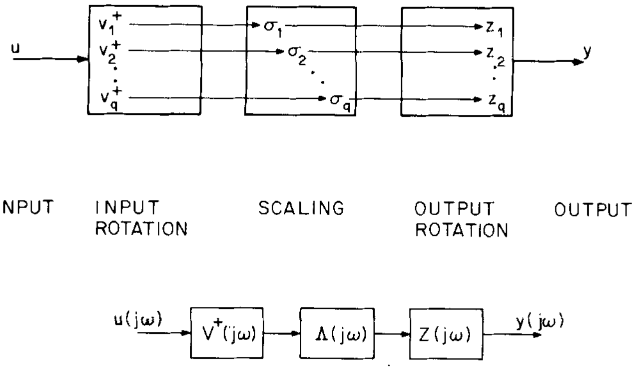


Figure 1. Geometric interpretation of the SVD. Transfer function matrix G is partitioned into three parts: input rotational matrix V^+, scaling matrix Λ, and output rotational matrix Z.

The above equations do not contain a $q + 1$ term because it is multiplied by a zero singular value. Thus, a singular value decomposition of the matrix $G(s)$ defines an input space spanned by a set of orthonormal basis vectors $\{v_i(s)\}_1^q$, an output space spanned by a set of orthonormal basis vectors $\{z_i(s)\}_1^q$, and a gain space defined by the set of singular values $\{\sigma_i(s)\}_1^q$. Furthermore, a one-to-one correspondence is established between these spaces as it illustrated in Figure 1.

It is now possible to interpret the transfer function matrix geometrically. An input vector in the direction of $v_i^+(s)$ propagates through the input space, is scaled by the gain $\sigma_i(s)$, and reappears in the output direction $z_i(s)$.

Some Interpretations of Process Control Properties Provided by Singular Value Analysis

The singular values give important information about the system. It is well-known in numerical analysis (Forsythe, 1977) that the condition number of a matrix is defined as

$$\gamma(s) \equiv \frac{\sigma^*(s)}{\sigma_*(s)} \tag{10}$$

where σ^* and σ_* denote the maximum and minimum singular values, respectively. The condition number quantifies the sensitivity of the system with respect to uncertainties in the matrix (modeling errors). Joseph and Brosilow (1978) use it to select measurement structure for inferential control. The magnitude of the minimum singular value, $\sigma_*(s)$, is a measure of the minimum distance to the nearest singular matrix and hence also a measure of the invertibility of the system. Thus $\sigma_*(s)$ discloses potential difficulties when implementing feedback control (Morari, 1982). The best control performance is achieved when $\sigma_*(s)$ is large.

Postlethwaite et al. (1981) have shown that for square systems σ_* and σ^* are the lower and upper bounds respectively for the moduli of the eigenvalues of $G(s)$. They also have shown that principal phases, which are defined as the arguments of the eigenvalues of the matrix ZV^+, under the condition that their spread is less than 180°, are the bounds for the arguments of the eigenvalues of $G(s)$. This technique is known as the *polar decomposition* of a matrix, in which a matrix can be rewritten as the product of two matrices, one containing the magnitude part, the other the phase part, just as a complex number can be expressed as the product of a magnitude and a phase. The usefulness of this polar representation lies in its clear characterization of phase information, an appealing idea and intuitively analogous to SISO concepts. Furthermore, this polar decomposition can be conveniently calculated from SVD and multivariable stability can be analyzed by constructing the principal region, which is a conic section defined by the maximum and minimum principal gains and phases.

Next we shall examine the effects which the basis vectors $\{z_i(s)\}_1^q$ and $\{v_i^+(s)\}_1^q$ have on the directionality within the system and the relation between interaction, singular values, and directionality.

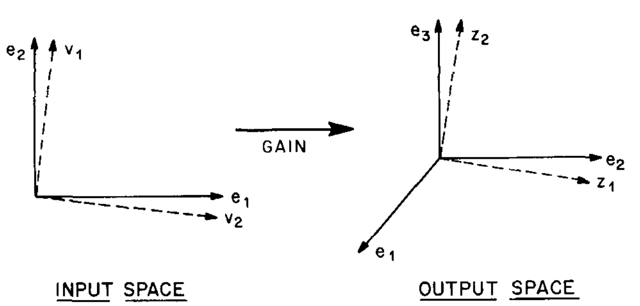


Figure 2. Pictorial representation of alignments between singular value vectors and standard basis vectors for a 3-output × 2-input system.

Relation between Input-Output Gain and Internal Structure of the System

Recall the input-output relation:

$$y(s) = G(s)\, u(s) \tag{4}$$

$G(s)$ may be expressed in terms of the singular values according to Eq. 8. Then

$$y(s) = \left[\sum_{1}^{q} \sigma_i(s) z_i(s) v_i^+(s)\right] u$$

By expanding $y(s)$ and $u(s)$ in the standard basis vectors – $\{e_k^m\}_{k=1}^m$ and $\{e_j^n\}_{j=1}^n$, where the superscripts refer to the vector dimension, the following equation is obtained:

$$\sum_{1}^{m} y_k(s) e_k^m = \sum_{1}^{q} \sigma_i(s)\, z_i(s)\, v_i^+(s) \left(\sum_{1}^{n} u_j(s)\, e_j^n\right)$$

Premultiplying this equation by e_k^{m+} and rearranging, we finally obtain:

$$\begin{aligned} y_k(s) &= \sum_{i=1}^{q} \sigma_i(s)\,(e_k^{m+}\; z_i(s)) \left(\sum_{j=1}^{n} u_j(s)(v_i^+(s) e_j^n)\right) \\ &= \sum_{i=1}^{q} \sigma_i(s) \sum_{j=1}^{n} u_j(s)\, \langle W_1(s), E_{k\ell} \rangle \end{aligned} \tag{11}$$

where $\langle W_i(s), E_{kj} \rangle \equiv (e_k^{m+} z_i(s))\,(v_i^+(s) e_j^n)$.

The product $\langle W_i(s), E_{jk} \rangle$ may be interpreted geometrically as a measure of the alignment of the singular decomposition vectors $z_i(s)$ and $v_i^+(s)$ to the standard basis vectors in the appropriate space, as is illustrated in Figure 2.

If $u_j(s) = 0$, for all $j \neq \ell$, Eq. 11 simplifies to

$$y_k(s) = \sum_{i=1}^{q} \sigma_i(s)\, \langle W_i(s), E_{k\ell} \rangle\, u_{\ell(s)} \tag{12}$$

and therefore

$$\frac{y_k(s)}{u_\ell(s)} = \sum_{i=1}^{q} \sigma_i(s)\, \langle W_i(s), E_{k\ell} \rangle = g_{k\ell}(s) \tag{13}$$

The equation expresses the gain between the kth output and the ℓth input in terms related to the internal structure of the system.

Definition of Natural Loop from the Dyadic Expansion

Suppose that for some i_o and for $s = j\omega$

$$|\langle W_{i_o}(j\omega), E_{k\ell} \rangle| \cong 1 \tag{15}$$

This implies that the i_o dyad is aligned closely with the basis dyad defined by the kth output and the ℓth input, or alternatively defined the (k,ℓ)th loop. Since the basis vectors $\{v_i(j\omega)\}_1^q$ and $\{z_i(j\omega)\}_1^q$ are orthonormal sets, we have

$$|\langle W_j(j\omega), E_{k\ell} \rangle| \cong 0 \qquad \forall_j \neq i_o \tag{16}$$

$$|\langle W_{i_o}(j\omega), E_{sp} \rangle| \cong 0 \qquad p \mp \ell, s \neq k \tag{17}$$

Thus we can conclude that except when the system is poorly conditioned, the (k,ℓ)th loop interacts minimally with other loops which we may select to control the system. Therefore we name the (k,ℓ)th loop a natural loop of the system.

Alignment Angle between System Dyad and Standard Basis Dyad

Let us formalize this directional analysis by starting with the following definition (also see Appendix 1):

$$\theta_{i_0} = \arccos |\langle W_{i_o}, E_{k\ell} \rangle| \tag{18}$$

where θ_{i_o} is the angle between the dyad, W_{i_o}, and the basis dyad $E_{k\ell}$. *When* θ_{i_o} is less than 15°, over 95% in magnitude of the i_o dyad comes from the (k,ℓ)th term and consequently the (k,ℓ)th loop is defined to have good directional property. Also θ_{i_o} is restricted between 0° and 90° by definition. The derivation of the interaction measure appears in Appendix 1.

Natural Loop Structure

If the system can be controlled adequately by natural loops, then we say that the system has a natural loop structure. Otherwise, compensation may be considered to obtain better directional behavior. We shall treat those cases below. Next we define an interaction measure.

Total Interaction Measure

The design procedure is essentially based on the direction of properties of each node, i.e. θ_{io}. However, to obtain a measure of how different the transfunction matrix G is from that of a completely decoupled system, we use the following total interaction measure:

$$\theta = \arccos \left[\left(\frac{\sum_{1}^{q} \sigma_i^2 \cos^2\theta_i}{\sum_{1}^{q} \sigma_i^2}\right)\right]^{1/2} \tag{19}$$

where the individual angles θ_i are defined by Eq. 18.

The physical interpretation for this total interaction measure is the ratio between the sums of the squares of the alignment of each dyad with a certain loop weighted by the appropriate squared singular values and the perfect alignment ($\cos\theta = 1$) weighted by the squared singular values. It can also be viewed as a geometric average of the contributions to the interaction by each node. If the maximum singular value is much greater than any other singular values and the corresponding alignment angle θ^* is of the same order of magnitude as the other alignment angles, then $\theta \cong \theta^*$. The choice of a total interaction measure is further justified in Appendix 1.

Graphical Representation of System Properties

By using singular value analysis and the angles defined above many system properties now are characterized by two plots similar to Bode diagrams. In Figure 4a the singular values and the directional angles are plotted as a function of frequency. In Figure 4b the condition number and the total interaction measure are plotted also as a function of frequency. Note that the condition number can be visualized by taking the distance between the maximum and minimum singular values in the first set of plots. The two sets are complementary because the former indicates the sensitivity and directionality of components in the system whereas the latter depicts total system properties.

Since singular values are multivariable counterparts of a SISO gain, it is not surprising to find that they have similar asymptotic behaviors as shown in Figure 4a. At low frequency the singular values remain constant; at high frequency they decrease linearly in a log-log representation.

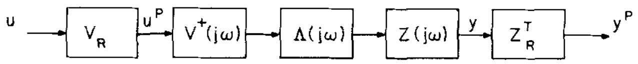


Figure 3. Block diagram showing decomposed system in the frequency domain with structural compensator.

The behavior of the directional angles is more difficult to predict as different patterns are observed for different systems.

In the next section we shall introduce a structural compensator which exploits the interaction within a system in order to deal with those systems which do not have a natural loop structure.

STRUCTURAL COMPENSATOR TO EXPLOIT SYSTEM INTERACTION

Very frequently we encounter a system which has no natural loop structure; therefore, compensation may improve the system's interaction. Among the methods reported in the literature are Rosenbrock's method (1974) for achieving dominance; static and dynamic decouplers (Luyben, 1970); and Hutchison's decoupling scheme (1980). These methods require an accurate model for the system; moreover, they are unable to quantify the performance of the decoupled system when modeling errors exist. In addition, the use of compensators which require extensive computations may be unjustifiable because of the system's uncertainties. In this section we develop a structural compensator based on SVD. Particularly attractive features of this method are that it is computationally simple and that it alerts the designer to the limitation of the compensator. Also, the analysis of the system behavior is carried out over a frequency domain such that we can design the compensator for the range of frequencies which affect the process.

A Natural Compensator Structure Suggested by SVD

To carry out the frequency analysis of the system's properties, we express the input-output relation as,

$$y(j\omega) = Z(j\omega)\ \Lambda(j\omega)\ V^+(j\omega)\ u(j\omega) \qquad (20)$$

Figure 1 is the block diagram of this relation. For a system which has no natural loop structure, some outputs are strongly affected by a number of inputs. For such an instance Figure 1 suggests a straightforward compensation structure where the matrices $Z^+(j\omega)$ and $V(j\omega)$ premultiply and postmultiply into Eq. 20, respectively. Therefore, the basis vector frames for the input space and the output space are rotated into noninteractive vector frames without altering the gain space's structure. Unfortunately, $Z^+(j\omega)$ and $V(j\omega)$ are complex matrices, so that compensation on a real system is not feasible. Instead, we use real vector approximations, V_R and Z_R, to these complex vector frames as shown in Figure 3. MacFarlane and Kouvaritakis (1977) have developed an elegant method to approximate the complex rotational matrices in the space of real matrices by requiring that the misalignment between the two matrix systems is minimal. They further showed that this minimization problem can be recast as the solution of two symmetric eigenvalue problems for which efficient numerical methods already exist. The calculation of the compensators is outlined in Appendix 2. We note from Figure 3 that the compensation method depends on the parameter ω. Thus, by judiciously selecting ω, compensation over the frequency band of interest can be achieved.

Further Interpretations of the Structural Compensator

An additional interesting feature revealed by Figure 3 is that the compensation method is equivalent to forming apparent inputs u^* from linear combinations of the inputs, and apparent outputs y^* from linear combinations of the outputs such that the apparent quantities when linked together properly are minimally interactive over a given frequency range. In essence, we exploit the interaction within the uncompensated system (Figure 1) to create a compensated system (Figure 3) with the appropriate directional property. Foss (1973), Morari (1981), and Ryskamp (1981) among others, have suggested that interaction could perhaps be exploited in such a manner. Morever, Niederlinski (1971) and Weischedel and McAvoy (1980) have reported instances where interactive systems indeed performed better. The present method provides a systematic and simple algorithm to exploit the natural interaction of the system.

In the subsequent section the elements of the SVD technique developed so far will be combined into a computer-aided analysis/synthesis scheme which forms the basis for the structural selection of control interconnections.

STRATEGIES FOR THE SYNTHESIS OF CONTROL STRUCTURES

Based on the theoretical analysis of the system we are now able to develop strategies for synthesis of control structures. Four different cases, encompassing the effects of modeling uncertainties and directional properties, can be identified.

Case 1. Good condition and good directional property.

$$\begin{array}{ll} \gamma(j\omega) < 10 & \omega\epsilon[\omega_1,\omega_n] \\ \theta_i(j\omega) < 15^\circ & i\epsilon[1,q] \end{array}$$

In this situation modeling uncertainties can be tolerated and a natural loop structure exists. The control loop structure will be the natural loop structure. The system's good condition number implies that the system will be well-behaved with the selected control structure for moderate modeling inaccuracies.

Case 2. Good condition and poor directional property.

$$\gamma(j\omega) < 10$$
$$\theta_i(j\omega) > 15^\circ$$

A natural loop structure does not exist. However, the structural compensator can be used to improve the directional property. The compensator is model-dependent since it relies on the transfer function $G(j\omega)$. However, the system's condition number is good so that compensation can be beneficial.

Case 3. Poor condition and good directional property.

$$\gamma(j\omega) > 10$$
$$\theta_i(j\omega) < 15^\circ$$

The system exhibits a good direction property but its condition number is poor. As a consequence, it is conceivable that small misalignments between the input and output spaces could be amplified because of the large differences in the magnitudes of the singular values. If, in fact, the transfer function $G(s)$ predicts the system's behavior accurately, then gain compensation can be used to improve the system's poor condition, i.e.,

$$G'(j\omega) = G(j\omega)K_g \qquad (21)$$

where the condition number of the compensated system $G'(j\omega)$ is of magnitude less than 10. Since the natural direction of the system is closely aligned with the standard basis vectors, the gain compensator K can be designed easily because each singular value of $G(j\omega)$ can be changed by the appropriate gain without altering the other singular values. However, the gain compensator's performance may be strongly affected by system perturbations. Also the magnitude of the gains which can be implemented on the system may be limited by physical restrictions such as control valve saturations.

Case 4. Poor condition and poor directional property.

$$\gamma(j\omega) > 10$$
$$\theta_i(j\omega) > 15^\circ$$

Theoretically, the structural compensator used in case 2 and the gain compensator used in case 3 can be combined to develop a control interconnection structure. However, both compensators are model-dependent and one should proceed with extreme caution or perhaps reconsider the process design.

In summary, this approach provides us with a diagnosis of the system. For the cases with good condition number, the strategy selected is less dependent on model accuracies. For the cases with poor condition, the control synthesis is more difficult. The designer has to be careful and may choose to consider additional modeling of the system and possibly adaptive strategy. In the worst case it may discourage the control attempt, calling for modifications in the process design or in the system configuration.

EXAMPLES

In this section we demonstrate our SVD approach on well-known examples from the literature which each represent a different approach to the interaction problem. Since we are primarily interested in the structural decision, we focus on that aspect and do not present closed-loop simulations. The original treatments of the examples include open- and closed-loop simulations to illustrate the effect of different loop selections. We shall demonstrate that the SVD approach leads to the same structural decisions as in the original works. The examples are discussed in increasing order of the complexity of the control problem.

Hutchison's Example

Hutchison (1979) developed a decoupler for multivariable systems to assure diagonal dominance. However, his methodology relies on function optimization which requires elaborate computations. Although the decoupler takes advantage of optimal feedback control, which insures some degree of robustness and sensitivity in the closed loop system, no direct analyses of those system properties were provided with the interaction analysis. To compare Hutchison's approach to the SVD analysis we consider the following example from Hutchison:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}; \qquad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The system is open-loop stable since all the poles are in the left hand plane. The dominant time constant is 1, so the relevant frequency band width for this system is around $\omega = 1$.

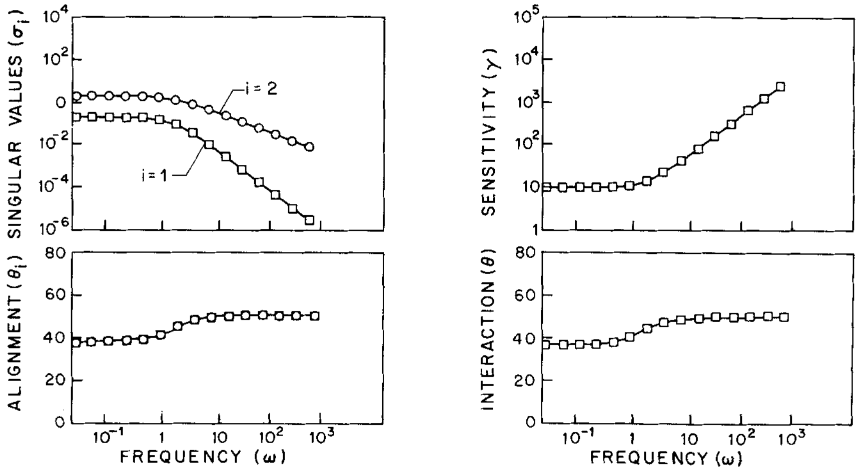
Figure 4a shows the singular values σ_i and the alignment angles θ_j for the individual modes as functions of the frequency w on a log-log plot, while Figure 4b gives the corresponding condition number, γ, and the total measure of interaction, θ, as functions of ω. We note that the two alignment angles for the 2X2 system are the same. The graphical representation is similar to that employed in Bode plots and the singular values are multivariable analogs to the gain in a SISO system. Therefore, it is not surprising to find that the σ-plots remain constant at low frequency and decrease linearly at high frequency. Furthermore, since the singular values are represented on a log-log scale, the distance between the σ^* curve and the σ_* curve is the logarithm of the condition number, γ, which increases at high frequencies as shown in Figure 4b. Consequently, the sensitivity of the system deteriorates at high frequency. Fortunately, the condition number has not increased significantly in the frequency range of interest around $\omega = 1$. Therefore the system has adequate sensitivity for us to attempt to design structural compensators.

The angle of alignment, θ_i, exceeds 15° for all frequencies, so the system has no natural loop structure and compensators must be introduced to reduce the interaction within the system. Figure 5 shows the system properties, σ, γ and θ, after structural compensation at $\omega = 1.0$ following the method outlined previously and in Appendix 2. Only the angle of alignment and the measure of interaction have been changed. Furthermore, since the original angle of alignment varies only slightly with frequency, the structural compensator virtually eliminates interaction over the entire frequency range. The subsequent examples will demonstrate that in more complex systems the compensation will be limited to a specified range of frequencies.

The present example demonstrates how SVD analysis yields structural information on the open-loop system and enables us to design a structural compensator to minimize loop interactions without changing the system sensitivity. Thus our approach decouples the interaction and sensitivity analysis so that they can be handled independently. Consequently, the feedback control system can easily be designed once the analysis and the proper compensations have been performed. Hutchison's original method, although very useful, requires that the interaction compensation and the control system design be done simultaneously, which is a major disadvantage.

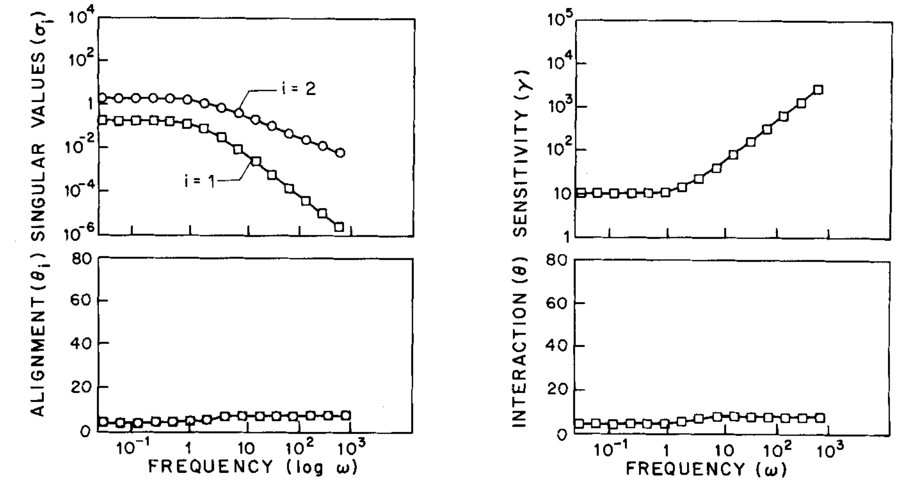
Tung's Numerical Example

As a second test of our methodology we consider a numerical example given by Tung and Edgar (1977). This example is interesting since Tung and Edgar have shown that static and dynamic decoupling procedures may lead to different structural decisions. The system matrices are:



(a) (b)

Figure 4. Hutchison's example without structural compensation: (a) plots of singular values and alignments; (b) plots of condition number and total interaction.



(a) (b)

Figure 5. Hutchison's example with structural compensation: (a) plots of singular values and alignments; (b) plots of condition number and total interaction. Compensator designed at ω = 1.

$$A = \begin{bmatrix} -3 & -5 & 0 \\ 1 & -5 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

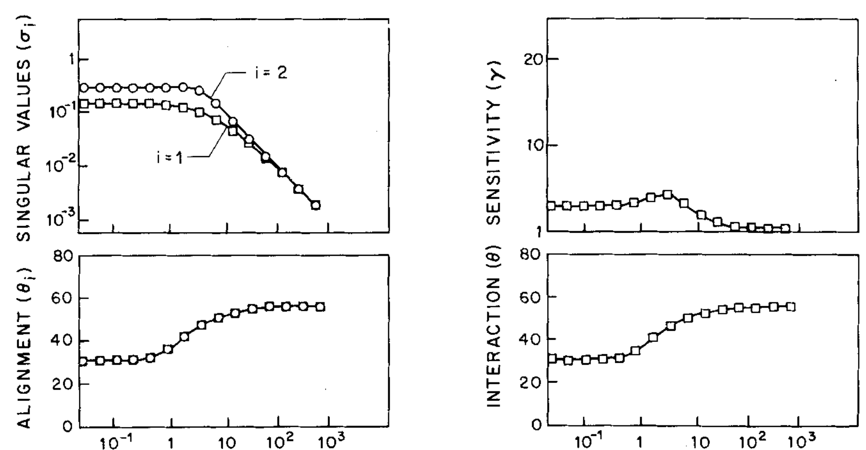
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The dominant time constant of the process is 3. By using their dynamic interaction measure, they concluded that even though at steady state u_1 and u_2, when linked with either y_1 or y_2, have the same measure of interaction, dynamically y_1 is more sensitive to u_1 and y_2 is more sensitive to u_2. Let us now apply the singular value decomposition.

Figure 6 illustrates the sensitivity, directionality, and robustness for this example. The minimum singular value is ~0.16 at low frequencies and in view of the low characteristic frequency 0.333 this indicates that large control efforts may be needed. However, the robustness, as measured by the condition number, is good since it remains less than 3 at low frequencies and approaches an asymptotic value of 1 at high frequencies. The most interesting behavior of this system is exhibited by the control loop structure. At low ω the loop structure selected by maximizing the amount of alignment between the system dyad and the standard basis dyad is $(y_1 - u_2)$ and $(y_2 - u_1)$. The 40° angle of alignment implies that this loop structure is preferred but clearly is not dominant. At high ω the loop structure is inverted in the sense that the $(y_1 - u_1)$ and $(y_2 - u_2)$ couplings have become the preferred loop structure. This confirms the conclusion that Tung and Edgar arrived at with their dynamic measure approach. A more subtle implication of this loop inversion is that the structural compensation will not work over the entire frequency range, as shown in Figure 7. In this case we can compensate either in the low frequency range or in the high frequency range, but not over the entire frequency domain. The explanation for this behavior is quite simple. The structural compensator is formed by redefining apparent inputs and outputs at a characteristic system frequency as linear combinations of the original inputs and outputs. Thus, if an output is sensitive to a certain input at low frequency and becomes more sensitive to a different input at high frequency, a compensator designed at low frequency will not give good alignments at high frequencies. In such instances a dynamic version of the structural compensator will be useful. This is currently under investigation.

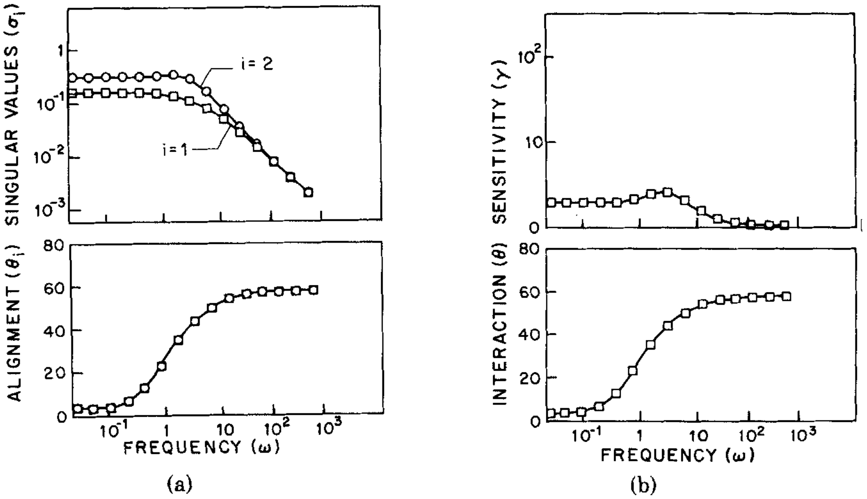
Control Structure Synthesis for a Distillation Column

Having tested our model on two numerical examples, we turn to the analysis of distillation column control, which poses challenging structural problems. A simple experimental model for a methanol distillation column is given by Wood and Berry (1973) who model the column with a 2X2 transfer function matrix in



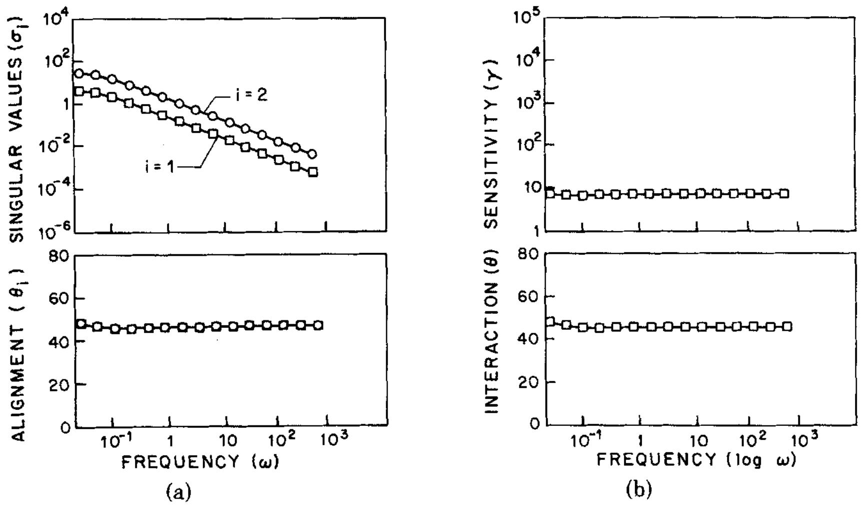
(a) (b)

Figure 6. Example of Tung and Edgar without structural compensation: (a) plots of singular values and alignments; (b) plots of condition number and total interaction.



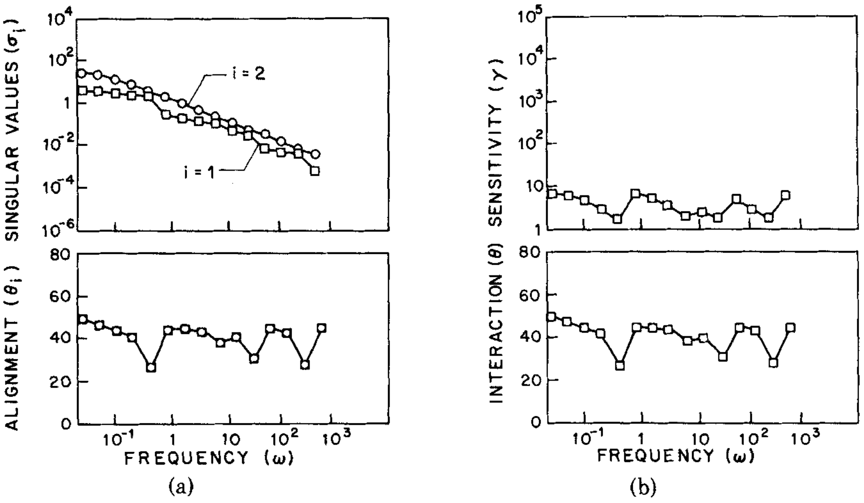
(a) (b)

Figure 7. Example of Tung and Edgar with structural compensation: (a) plots of singular values and alignments; (b) plots of condition number and total interaction. Compensator designed at ω = 0.333.



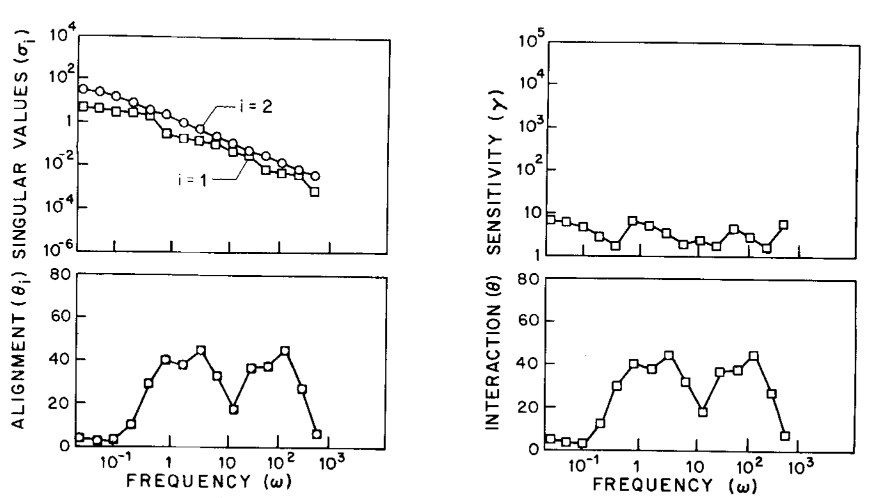
(a) (b)

Figure 8. Distillation system of Wood and Berry without multiple time delays and structural compensator: (a) plots of singular values and alignment; (b) plots of condition number and total interaction.



(a) (b)

Figure 9. Distillation system of Wood and Berry with multiple time delays and without structural compensator: (a) plots of singular values and alignment, (b) plots of condition number and total interaction.



(a) (b)

Figure 10. Distillation system of Wood and Berry with multiple time delays and structural compensator: (a) plots of singular values and alignment; (b) plots of condition number and total interaction. Compensator designed at $\omega = 0.05$.

which each element of the matrix is a first-order system with a time lag:

$$\begin{bmatrix} \text{MeOH}_{\text{top}} \\ \text{MeOH}_{\text{bottom}} \end{bmatrix} = \begin{bmatrix} \dfrac{12.8\,e^{-s}}{16.7s+1} & \dfrac{-18.9\,e^{-3s}}{21.0s+1} \\ \dfrac{6.6\,e^{-7s}}{10.9s+1} & \dfrac{-19.4\,e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R_{\text{reflux}} \\ R_{\text{steam}} \end{bmatrix} \tag{22}$$

The dominant process time constant, discounting the time lags for now, is approximately 20. Thus the relevant frequency band width for this system is around frequency $\omega = 0.05$.

Figure 8 shows the condition number, γ, the singular values, σ, and the angles of alignment determined from the SVD of the transfer $G^*(j\omega)$, where G^* represents G without time delays. The condition number is quite good, but the alignment is poor. Therefore, there is no natural loop structure and we would have to construct a compensator. However, let us first consider the additional effects of the time delays.

Figure 9 shows the γ, σ, and θ plots for the complete system with time delays. A comparison of Figure 8 and 9 reveals that the time delays have perturbed each plot and in particular have introduced large fluctuations in the alignment. Again we find that the sensitivity is adequate but the natural alignment is poor. Therefore, we proceed to design a structural compensator for the system to attempt to improve the directionality of the system. However, as illustrated in Figure 10 the compensation is only successful at low frequencies. In fact, the compensated system exhibits worse fluctuations than the original system. This is to be expected since the compensator defines new apparent inputs and outputs from linear combinations of the original input and outputs such that instead of having one characteristic time delay for each input-output relation there is a multimode of delays for each apparent input-output relation.

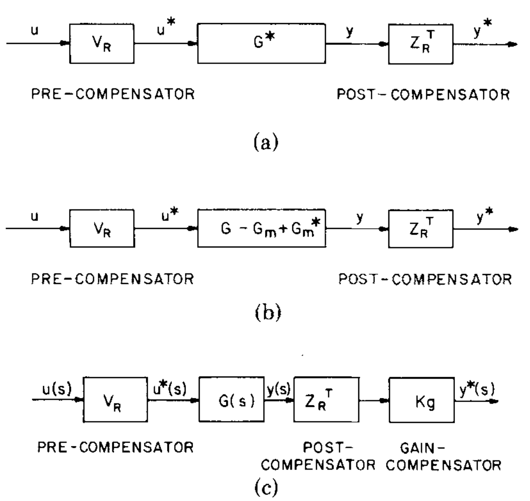


Figure 11. (a) Block diagram of a system with structural compensation to improve alignment and perfect multivariable Smith predictor to eliminate multiple time delays. (b) Block diagram of a system with structural compensation and model mismatch in multivariable Smith predictor. (c) Block diagram of a system with both structural and gain compensators.

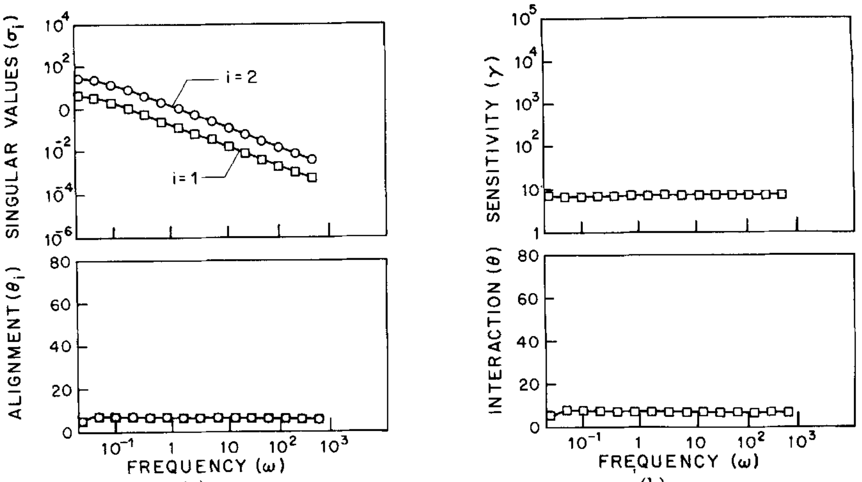
To eliminate the effects of the time delays we include the multivariable Smith Predictor developed by Ogunnaike and Ray (1979). The block diagram for this combined scheme is shown in Figure 11a. In the case of a perfect model the system becomes equivalent to structural compensation on G^*. It is then straightforward to design a compensator around the characteristic frequency $\omega = 0.05$. The results of this combined method are shown in Figure 12, which demonstrates that the directionality is vastly improved without changes in the sensitivity of the system.

Realistically, modeling errors are unavoidable in any physical situation. Our assumption of a perfect model in the above example serves mainly to illustrate the potential of the approach. Therefore it is important to analyze the effects of model mismatch on the combined compensator and predictor system. In the case of model mismatch the system being compensated is not G^*, but $G - G_m + G_m^*$ (Figure 11b), which does have time-lag terms. Therefore, as shown in Figure 13 for a case of 5% mismatch of the delays, fluctuations reappear in the angle of alignment, in particular at high frequencies. Nevertheless, at low frequency ($\omega < 5$) the structural compensator and the predictor perform quite well. Since the dominant time constant for this system is 20, corresponding to $\omega_{cr} = 0.05$, we conclude that the combined compensator and predictor scheme works adequately in the frequency range of interest. Moreover, we also note from this analysis that a system which has a larger dominant time constant (slow response) can tolerate model mismatch better than a system which has a smaller dominant time constant (fast response), as one would expect.

Distillation Column with Sidestreams

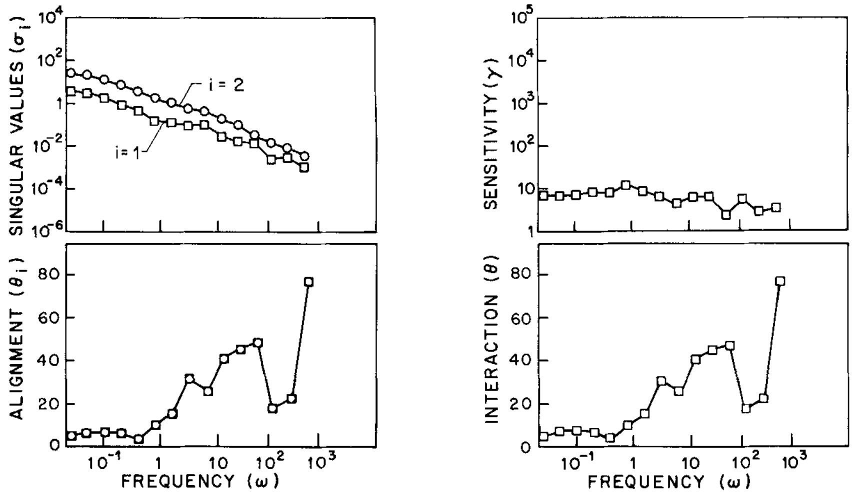
As a fourth and final example we consider a distillation column with sidestreams modeled by a 3X3 transfer function matrix (Ray, 1981)

$$G = \begin{bmatrix} \dfrac{0.7}{1+9s} & 0 & 0 \\ \dfrac{2.0}{1+8s} & \dfrac{0.4}{1+6s} & 0 \\ \dfrac{2.3}{1+10s} & \dfrac{2.3}{1+8s} & \dfrac{2.1}{1+7s} \end{bmatrix}$$



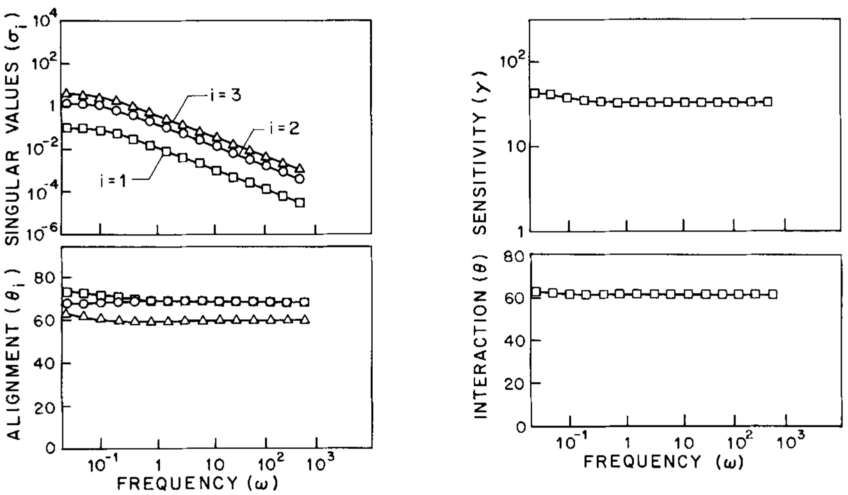
(a) (b)

Figure 12. The distillation system of Wood and Berry with structural compensation and perfect multivariable Smith predictor: (a) plots of singular values and alignment; (b) plots of condition number and total interaction.



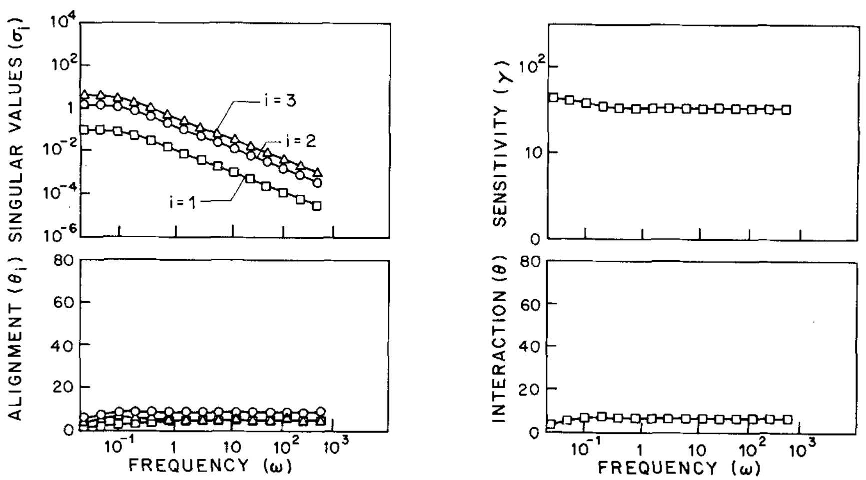
(a) (b)

Figure 13. The distillation system of Wood and Berry with structural compensation and model mismatch in multivariable Smith predictor: (a) plots of singular values and alignment; (b) plots of condition number and total interaction.



(a) (b)

Figure 14. A distillation system with sidestream and without structural compensation: (a) plots of singular values and alignment; (b) plots of condition number and total interaction.



(a) (b)

Figure 15. A distillation system with sidestream and structural compensation: (a) Plots of singular values and alignment; (b) plots of condition number and total interaction.

The manipulated variables are the reflux rate, u_1, and the amounts drawn off the sidestreams, u_2 and u_3, while the measured variables are the concentrations of the top and sidestreams, y_1, y_2, and y_3. Ray has shown that if three single-feedback loops are used to control the system, the only feasible loop structure is:

$$y_1 - u_1$$
$$y_2 - u_2$$
$$y_3 - u_3$$

Any other structure will lead to controlability and observability problems due to the special structure of the transfer function matrix. Ray also pointed out that considerable interactions exist. By analyzing the singular value decomposition of $G(j\bar{\omega})$, we conclude that the system has poor sensitivity ($\gamma = 100.0$ and the loop structure is very poorly aligned as illustrated in Figure 14. Moreover, large control efforts will be needed in this feedback control scheme since the minimum singular value is small (<0.1). Thus, there are severe control problems associated with the single loop feedback scheme.

Having designed a well aligned system by using the structural compensator (Figure 15), we can include a gain compensator in a straightforward manner to improve the system's sensitivity as illustrated in Figure 11c along with the entries to the gain matrix, K_g. These entries are limited by the maximum gain the system can tolerate without loss of stability or saturation of control elements. In addition, we must be careful when implementing these compensators since they are designed around a poor-condition mode.

The above examples illustrate the systematic approach to control structure synthesis that is provided by singular value decomposition and the related quantitative measures of interaction and sensitivity. In addition, they show how a designer would be able to identify elements of the process model which could significantly influence the performance of the control scheme such that the modeling effort could be concentrated around dynamically important process elements.

ACKNOWLEDGMENT

This research was supported in part by NSF grant No. ENG75-11165-A01 and by a Shell Faculty Career Initiation Award. The authors also wish to thank the University of Minnesota Computer Center for its support.

APPENDIX 1: MEASURE OF INTERACTION

Consider the singular value spectral representation of a $m\times n$ open loop matrix transfer function $G(s)$ whose rank is q.

$$G(s) = \sum_{i=1}^{q} \sigma_i(s) W_i(s), \qquad \sigma_i > 0 \qquad \text{(A1)}$$

where $\{\sigma_i\}$ are the singular values of G and $\{W_i\}$ are the dyadic expansion matrices defined in terms of singular vectors as follows

$$W_i(s) = z_i(s) v_i^+(s)$$

Through expansion (Eq. A1) the system transfer function is expressed as a linear combination of the q nodal contributions. Each of the nodal terms consists of a scaling factor σ_i, and a rotational transformation W_i.

For simplicity the argument (s) will be dropped from the notation. To illustrate the loop selection procedure we suppose that G is a 2X2 diagonal matrix

$$G = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \qquad g_1, g_2 > 0$$

The singular value expansion yields

$$G = g_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + g_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(A2)}$$

Since the matrix G does not induce input-output interaction, the rotational matrices have zero entries except at one position. Obviously the input-output pairing is $[u_1 - y_1, u_2 - y_2]$.

Now let us assume G is given by

$$G = \begin{bmatrix} -.334 & -1.063 \\ 4.931 & 0.677 \end{bmatrix}$$

The singular value expansion yields

$$G = 5 \begin{bmatrix} -.098 & -0.016 \\ .983 & 0.156 \end{bmatrix} + 1 \begin{bmatrix} .156 & -.983 \\ 0.016 & -0.098 \end{bmatrix} \qquad \text{(A3)}$$

Here node 1 introduces the largest scaling (gain) factor. Examination of the first rotational matrix shows that the entry w_{21} has the longest absolute value. Therefore, the first node has a natural loop, the pair $[u_1 - y_2]$. Similarly we can conclude that the natural loop for the second node is $[u_2 - y_1]$.

From these two examples we may extract the following conclusions:

(i) Each node has an associated rotational matrix. Its maximum entry $(\mathrm{Max}|x_{ij}|_\ell)$ defines a $u_i - y_j$ pairing (loop).

(ii) A measure of interaction for the ℓth node should quantify the "difference" between the ℓth rotational matrix and a matrix of the same dimensions associated with the ideal natural loop. The latter matrix has zero elements except one located at the i-jth position defined by the u_i-y_j perfect pairing.

(iii) To obtain a measure of the total interaction the nodal measures of interactions should be weighted by their corresponding singular value.

To quantify the interaction associated with the rotational matrix $\boldsymbol{W}_i$ we define a space H with $m \times n$ matrices as elements. The entries of the matrices are complex numbers. The space is equipped with an inner product and a norm defined as follows:

$$\langle X,Y \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} (x_{ij} y_{ij}^{+}), \qquad x_{ij},\, y_{ij} \in C^{m \times n}$$

$$\| X \| = \langle X,X \rangle^{1/2}$$

where $(\cdot)^{+}$ denotes complex conjugation.

Let L be a space of uncoupled $\boldsymbol{D}_{ij}$ matrices defined as

$$\boldsymbol{D}_{ij} = \begin{bmatrix} d_{11} - d_{1n} \\ : \quad : \ddots \\ d_{m1} - d_{mn} \end{bmatrix}, \qquad d_{k\ell} = \begin{cases} 1, & (i,j) = (k,\ell) \\ 0, & (i,j) \neq (k,\ell) \end{cases}$$

In other words, $\boldsymbol{D}_{ij}$ is the rotational matrix associated with the $u_i - y_j$ pairing in the ideal case of natural perfect loop structure (see Eq. A2).

Since H is a finite dimensional normal linear space, the subspace L is closed (Naylor and Sell, 1982). Hence the node $\boldsymbol{W}_\ell$ can be orthogonally decomposed into an interactive and noninteractive part

$$\boldsymbol{W}_\ell = \overline{\boldsymbol{W}}_\ell + \tilde{\boldsymbol{W}}_\ell, \qquad \overline{\boldsymbol{W}}_\ell \perp \tilde{\boldsymbol{W}}_\ell, \qquad \overline{\boldsymbol{W}}_\ell \in L$$

For the ℓth node we know that the $\text{Max}_{ij} |x_{ij}|$ defines the $i - j$ pairing. Hence, the projection of X into the uncoupled space L is given by

$$\overline{\boldsymbol{W}}_\ell = \boldsymbol{D}_{ij} \langle \boldsymbol{W}_\ell, \boldsymbol{D}_{ij} \rangle = \begin{bmatrix} 0 - - - 0 \\ : - (w_{ij})_\ell - : \\ 0 - - - 0 \end{bmatrix}$$

Then the generalized angle between the dyad $\boldsymbol{W}_\ell$ and its orthogonal projection into the subspace of decoupled systems is given by

$$\cos \theta_\ell = \frac{\| \overline{\boldsymbol{W}}_\ell \|}{\| \boldsymbol{W}_\ell \|}$$

Thus, the angle θ_ℓ measures the misalignment induced by the ℓth node with respect to the ideal natural loop. To evaluate the denominator we apply the definition of the norm:

$$\| \boldsymbol{W}_\ell \|^2 = \langle W, W \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} (w_{ij})_\ell (w_{ij}^{+})_\ell$$

which may be written in terms of the singular decomposition vectors, $\boldsymbol{z}$ and $\boldsymbol{v}$, as:

$$\| \mathbf{W}_\ell \|^2 = \sum_{i=1}^{m} (z_i z_i^{+})_\ell \left[\sum_{j=1}^{n} (v_j v_j^{+})_\ell \right]$$

Since $\boldsymbol{z}$ and $\boldsymbol{v}$ are orthonormal it then follows that

$$\| \mathbf{W}_\ell \| = 1$$

and the angle expression reduces to:

$$\cos\theta_\ell = \sqrt{w_{ij} w_{ij}^{+}} \tag{A4}$$

ij = the input-output pairing associated with the ℓth node.

Thus: if $\theta = 0°$ perfect alignment, no interaction

if $\theta = 90°$ no alignment, orthogonal

As total measure of interaction we use the following weighted average over the nodal measures of alignment

$$\cos \theta = \left[\frac{\sum_{i=1}^{q} \sigma_i^2 \cos^2 \theta_i}{\sum_{i=1}^{q} \sigma_i^2} \right]^{1/2} \tag{A5}$$

This definition can be justified as an upper bound for the total interaction of the matrix $\boldsymbol{G}$. Suppose that the $q(u_i - y_j)$ pairings have been selected and let DCH be a subspace of decoupled matrices. The matrix G can then be orthogonally decomposed into an interactive and a noninteractive part:

$$\boldsymbol{G} = \overline{\boldsymbol{G}} + \tilde{\boldsymbol{G}}, \qquad \overline{\boldsymbol{G}} \perp \tilde{\boldsymbol{G}}, \qquad \tilde{\boldsymbol{G}} \in D$$

The global interaction measure is then:

$$\cos\phi = \frac{\| \overline{\boldsymbol{G}} \|}{\| \boldsymbol{G} \|}$$

The denominator may be expanded in terms of the orthonormal basis $\{W_\ell\}$. Thus, by using Eq. 9 we obtain

$$\| \boldsymbol{G} \|^2 = \sum_{j=1}^{q} \sum_{i=1}^{q} \sigma_i \sigma_j \langle \boldsymbol{W}_i, \boldsymbol{W}_j \rangle = \sum_{i=1}^{q} \sigma_i^2 \tag{A6}$$

On the other hand, the numerator can be expanded as:

$$\| \overline{\boldsymbol{G}} \|^2 = \sum_{\ell=1}^{q} \sigma_\ell^2 \, |w_{ij(\ell)}|_\ell^2 + \sum_{\substack{p,k,\ell \\ p \neq k, r \neq \ell}}^{q} \sigma_k \sigma_\ell [w_{ij(p)}]^k [w_{ij(p)}^{+}]_\ell$$

where $[W_{ij(p)}]_\ell$ denotes the ij entry of $\boldsymbol{W}_\ell$ associated with the pth loop. By combining eqs. A4, A5, and A6, we find that the interaction measure then can be expressed as:

$$\cos^2\phi = \cos^2\theta + \epsilon \tag{A7}$$

with

$$\epsilon = \left(\sum_{\substack{p,k,\ell \\ p \neq k, p \neq \ell}}^{q} \sigma_k \sigma_\ell [w_{ij(p)}]_k [w_{ij(p)}^{+}]_\ell \right) \left(\sum_{\ell=1}^{q} \sigma_{\ell=1}^2 \, \sigma_\ell^2 \right)^{-1}$$

For an ideal loop structure $[w_{ij(k)}]_\ell = 0$ for $k \neq \ell$ and thus, $\epsilon = 0$. Furthermore, application of the triangle inequality leads to

$$\cos^2\phi \geqslant \cos^2\theta.$$

Hence, the measure θ is conservative and becomes equal to ϕ in the limit of a natural loop structure.

APPENDIX 2 (MacFarlane and Kouvaritakis, 1977)

Given the set of decomposition vectors $\{z_i(j\omega), i=1,2,\ldots,m\}$ we seek a set of real vectors $\{a_i, i=1,2,\ldots m\}$ to approximate the $z_i(j\omega)$'s at some ω. We take as a measure of this approximation the misalignment between z_i and a_i by defining

$$\phi_i = \frac{|(z_i, a_i)|^2}{\sum_{\substack{j=1 \\ j \neq i}}^{m} |(z_j, a_i)|^2} \tag{A2.1}$$

Rewriting $z_j = \alpha_j + j\beta_j$ one can show that

$$\phi_i = \frac{a_i^{+} \; C_i \, a_i}{a_i^{+} \; D_i \; a_i} \tag{A2.2}$$

where

$$C_i \equiv \alpha_i \alpha_i^{+} + \beta_i \beta_i^{+}$$

$$D_i = \sum_{\substack{j=1 \\ j \neq i}}^{m} (\alpha_j \alpha_j^{+} + \beta_j \beta_j^{+})$$

From the maximum principle it must be true that

$$C_i \, a_i = \phi_i \, D_i \, a_i \tag{A2.3}$$

Eq. A2.3 represents a generalized eigenvalue problem. Therefore ϕ_i must be the maximum generalized eigenvalue and a_i the corresponding generalized eigenvector. Further algebraic manipulations yield

$$a_i = R_d \, \Lambda_d^{-1} \, v_i \tag{A2.4}$$

where R_d and Λ_d are defined by the spectral decomposition of D_i a real positive definite matrix,

$$D_i = R_d \, \Lambda_d^2 R_d^t \tag{A2.5}$$

and v_i is the eigenvector corresponding to the maximum eigenvalue of the matrix S_i

$$S_i = \Lambda_d^{-1} R_d^t C_i R_d \Lambda_d^{-1} \tag{A2.6}$$

Thus this generalized eigenvalue problem has been transformed into a problem of determining the spectral decomposition of $\boldsymbol{D}_i$ and S_i, which both are real symmetric matrices. By repeating this calculation for different choices of i we build the structural compensator whose columns consist of the real vectors $\boldsymbol{a}_i$.

NOTATION

x = state vector
y = output vector
u = input vector
y^* = apparent output vector
u^* = apparent input vector
y_k = kth output
u_j = jth input
z_i = ith left singular value decomposition vector
v_i = ith right singular value decomposition vector
Z = left singular value decomposition matrix
V = right singular value decomposition matrix
$\boldsymbol{A}$ = state transition matrix
$\boldsymbol{B}$ = input matrix
$\boldsymbol{C}$ = measurement matrix
$\boldsymbol{G}$ = transfer function matrix
$\boldsymbol{W}_i$ = ith dyad in expansion for G
$\boldsymbol{Z}_R$ = left structural compensator
$\boldsymbol{V}_R$ = right structural compensator
$\boldsymbol{K}_g$ = gain compensator
q = rank of G
m = dimension of measurement vector y
n = dimension of input vector u
r = dimension of state vector x
e_k^m = kth standard basis vector in R^m
e_j^n = jth standard basis vector in R^n

Greek Letters

Λ = diagonal matrix containing singular values of G
ω = frequency
ω_1 = lower bound on frequency of interest
ω_n = upper bound on frequency of interest
σ_i = ith singular value
σ^* = maximum singular value
σ_* = minimum singular value
γ = condition number
θ_i = alignment angle of ith dyad
θ = total interaction measure
θ^* = alignment angle corresponding to σ^*

Superscripts

$+$ = transpose and conjugate

LITERATURE CITED

Bristol, E. H., "On a New Measure of Interaction for Multivariable Process Control," *IEEE Trans. Auto. Contr.*, **AC-11** (7), 133 (1966).

Cruz, J. B., J. S. Freudenberg, and D. P. Looze, "A Relationship between Sensitivity and Stability of Multivariable Feedback System," *IEEE Trans. Auto. Contr.*, **AC-26** (1), 66 (1981).

Doyle, J. C. and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Trans. Auto. Contr.*, **AC-26** (1), 4 (1981).

Forsythe, G. E., M. A. Malcolm, and C. B. Moler, *Computer Methods for Mathematical Computations*, Prentice Hall, Englewood Cliffs, NJ (1977).

Foss, A. S., "Critique of Chemical Process Control," *AIChE J.*, **19**, 209 (1973).

Garcia, C. E., and M. Morari, "Internal Model Control—A Unifying Review and Some New Results," *IEC Proc. Des. Dev.*, **21**, 308 (1982).

Gagnepain, J. P., and D. E. Seborg, "Analysis of Process Interactions with Applications to Multiloop Control System Design," *Ind. Eng. Chem. Proc. Des. Dev.*, **21**, 5 (1982).

Hutchison, W. J., "Decoupling of Multivariable Systems by Function Optimization," *Int. J. Contr.*, **30**, 1,001 (1979).

Joseph, B., and C. Brosilow, "Inferential Control of Processes. I: Steady State Analysis and Design," *AIChE J.*, **24**, 485 (1978).

Lau, H., "Studies on Control of Integrated Chemical Processes," Ph.D. Dissertation, Univ Minnesota (1982).

Lee, W., and W. V. Weekman, Jr., "Advanced Control Practice in the Chemical Process Industry: A View from Industry," *AIChE J.*, **22**, 27 (1976).

Luyben, W. L., "Distillation Decoupling," *AIChE J.*, **16**, 198 (1970).

MacFarlane, A. G. J., and B. Kouvaritakis, "A Design Technique for Linear Multivariable Feedback Systems," *Int. J. Contr.*, **25**, 837 (1977).

Morari, M., "Integrated Plant Control: A Solution at Hand or a Research Topic for the Next Decade," *Proc. Chem. Proc. Control II*, Seborg and Edgar, Eds. (1981).

Morari, M., "Flexibility and Resiliency of Process Systems," *Proc. Int. Symp. Proc. Sys. Eng.*, Kyoto, Japan (1982).

Naylor, A. W., and G. R. Sell, *Linear Operator Theory in Engineering and Science*, Springer-Verlag, New York (1982).

Niederlinski, A., "Two-Variable Distillation Control: Decouple or Not Decouple," *AIChE J.*, **17**, 1,261 (1971).

Noble, B., and J. W. Daniel, *Applied Linear Algebra*, Prentice-Hall, Englewood Cliffs, NJ (1977).

Ogunnaike, B. A., and W. H. Ray, "Multivariable Controller Design for Linear Systems Having Multiple Time Delays," *AIChE J.*, **25**, 1,043 (1979).

Postlethwaite, I., J. M. Edmunds, and A. G. J. MacFarlane, "Principal Gains and Principal Phases in the Analysis of Linear Multivariable Feedback Systems," *IEEE Trans. Auto. Contr.*, **AC-26**, (1), 32 (1981).

Ray, W. H., *Advanced Process Control*, McGraw Hill, New York (1981).

———, "Multivariable Process Control—A Survey," *Proc. Int. Symp. Proc. Syst. Eng.*, Kyoto, Japan (1982).

Rosenbrock, H. H., *Computer-Aided Control System Design*, Academic Press, New York (1974).

Ryskamp, C. J. "Explicit vs. Implicit Decoupling in Distillation Control," *Proc. Chem. Proc. Control II*, Seborg and Edgar, Eds. (1981).

Safanov, M. G., A. I. Laub, and G. L. Hartmann, "Feedback Properties of Multivariable Systems: The Role and Use of Return Difference Matrix," *IEEE Trans. Auto. Contr.*, **AC-26** (1), 47 (1981).

Smith, C. R., C. F. Moore, and D. D. Bruns, "A Structural Framework for Multivariable Control Applications," *JACC* (1981a).

Smith, C. R., "Multivariable Process Control Using Singular Value Decomposition," Ph.D. Thesis, Univ. Tennessee (1981b).

Stephanopoulos, G., "Synthesis of Control Systems for Chemical Plants—A Challenge for Creativity," *Proc. Int. Symp. Proc. Syst. Eng.*, Kyoto, Japan (1982).

Tung, L. S., and T. F. Edgar, "Dynamic Interaction Analysis and its Application to Distillation Column Control," *IEEE Decision and Control Conf.*, 1,107 (1977).

Weber, R., and C. Brosilow, "The Use of Secondary Measurements to Improve Control," *AIChE J.*, **18**, 614 (1972).

Weischedel, K., and T. J. McAvoy, "Feasibility of Decoupling in Conventionally Controlled Distillation Columns," *Ind. Eng. Chem. Fund.*, **19**, 379 (1980).

Witcher, M. F., and T. J. McAvoy, "Interacting Control Systems: Steady State and Dynamic Measurement of Interaction," *ISA Trans.*, **16** (3), 35 (1977).

Wood, R. K., and M. W. Berry, "Terminal Composition Control of a Binary Distillation Column," *Chem. Eng. Sci.*, **28**, 1,707 (1973).

Manuscript received Oct. 5, 1982; revision received Oct. 19, 1983, and accepted Oct. 20.